

Summary of Important Definitions

May 11th

1 Lattice Theory

Definition 1.1. A partial order \preceq is a relation over a set S such that for every triple of elements $x, y, z \in S$ the following hold

- (reflexivity) $x \preceq x$
- (antisymmetry) $(x \preceq y \wedge y \preceq x) \Rightarrow x = y$
- (transitivity) $(x \preceq y \wedge y \preceq z) \Rightarrow (x \preceq z)$

Definition 1.2. Given a partial order \preceq over a set S and a subset $X \subseteq S$, a lower bound of X (resp. an upper bound of X) is an element $x \in S$ (Note that it may be the case that $x \notin X$) such that

- $\forall y \in X, \boxed{x} \preceq y$
- (resp. $\forall y \in X, \boxed{y} \preceq x$)

The greatest lower bound of a set $X \subseteq S$ (denoted as $\mathbf{glb}(X)$) is a unique upperbound of the set of all lowerbounds of X . The least upper bound of a set $X \subseteq S$ (denoted as $\mathbf{lub}(X)$) is a unique lowerbound of the set of all upperbounds of X . In general, $\mathbf{lub}(X)$ and $\mathbf{glb}(X)$ may not exist.

Definition 1.3. A (complete) lattice $\langle \mathcal{L}, \preceq \rangle$ is a set of elements \mathcal{L} and a partial order \preceq over \mathcal{L} such that for any set $S \subseteq \mathcal{L}$

- $\mathbf{lub}(X)$ and $\mathbf{glb}(X)$ exist and are unique.

1.1 Fixpoint Operators

Definition 1.4. Given a complete lattice $\langle \mathcal{L}, \preceq \rangle$, a fixpoint operator over the lattice is a function $\Phi : \mathcal{L} \rightarrow \mathcal{L}$ that is \preceq -monotone

- Φ is \preceq -monotone if for all $x, y \in \mathcal{L}$

$$x \preceq y \Rightarrow \Phi(x) \preceq \Phi(y)$$

(Note: this does not mean the function is inflationary, i.e., $x \preceq \Phi(x)$ may not hold)

A fixpoint is an element $x \in \mathcal{L}$ s.t. $\Phi(x) = x$.

Theorem 1.1 (Knaster-Tarski). *The set of all fixpoints of a fixpoint operator Φ on a complete lattice is itself a complete lattice. The least element of this new lattice exists and is called the least fixpoint (denoted as $\mathbf{lfp} \Phi$)*

Some intuitions about lattices

- The entire lattice has a biggest element $\mathbf{lub}(\mathcal{L}) = \top$ and a smallest element $\mathbf{glb}(\mathcal{L}) = \perp$
- When a lattice has a finite height (or finite domain). The least fixed point of a fixpoint operator can be computed by iteratively applying the fixpoint operator to \perp
- An operator may return an element that is not comparable to the input, however, after a comparable element is returned (either greater or less than) that comparability and direction are maintained for all subsequent iterations.
- Further, because \perp is less than all elements in the lattice, it is always the case that $\perp \preceq \Phi(\perp)$

2 Partial Stable Model Semantics

Definition 2.1. *A (ground and normal) answer set program \mathcal{P} is a set of rules where each rule r is of the form*

$$h \leftarrow a_0, a_1, \dots, a_n, \mathbf{not} b_0, \mathbf{not} b_1, \dots, \mathbf{not} b_k$$

where we define the following shorthand for a rule $r \in \mathcal{P}$

$$\begin{aligned} \text{head}(r) &= h \\ \text{body}^+(r) &= \{a_0, a_1, \dots, a_n\} \\ \text{body}^-(r) &= \{b_0, b_1, \dots, b_k\} \end{aligned}$$

Definition 2.2. *A two-valued interpretation I of a program \mathcal{P} is a set of atoms that appear in \mathcal{P} .*

Definition 2.3. *An interpretation I is a model of a program \mathcal{P} if for each rule $r \in \mathcal{P}$*

- *If $\text{body}^+(r) \subseteq I$ and $\text{body}^-(r) \cap I = \emptyset$ then $\text{head}(r) \in I$.*

Definition 2.4. *An interpretation I is a stable model of a program \mathcal{P} if I is a model of \mathcal{P} **and** for every interpretation $I' \subseteq I$ there exists a rule $r \in \mathcal{P}$ such that*

- *$\text{body}^+(r) \subseteq I'$, $\text{body}^-(r) \cap I \neq \emptyset$ (Note that this is I and not I') and $\text{head}(r) \notin I'$*

Definition 2.5. A three-valued interpretation (T, P) of a program \mathcal{P} is a pair of sets of atoms such that $T \subseteq P$. The $\boxed{\text{truth-ordering}}$ respects $\mathbf{f} < \mathbf{u} < \mathbf{t}$ and is defined for two three-valued interpretations (T, P) and (X, Y) as follows.

$$(T, P) \preceq_t (X, Y) \text{ iff } T \subseteq X \wedge P \subseteq Y$$

The $\boxed{\text{precision-ordering}}$ respects the partial order $\mathbf{u} < \mathbf{t}$, $\mathbf{u} < \mathbf{f}$ and is defined for two three-valued interpretations (T, P) and (X, Y) as follows.

$$(T, P) \preceq_p (X, Y) \text{ iff } T \subseteq X \wedge \boxed{Y \subseteq P}$$

Definition 2.6. A three-valued interpretation (T, P) is a model of a program \mathcal{P} if for each rule $r \in \mathcal{P}$

- $\text{body}(r) \subseteq P \wedge \text{body}^-(r) \cap T = \emptyset$ implies $\text{head}(r) \in P$, and
- $\text{body}(r) \subseteq T \wedge \text{body}^-(r) \cap P = \emptyset$ implies $\text{head}(r) \in T$.

Definition 2.7. A three-valued interpretation (T, P) is a stable model of a program \mathcal{P} if it is a model of \mathcal{P} and if for every three-valued interpretation (X, Y) such that $(X, Y) \preceq_t (T, P)$ there exists a rule $r \in \mathcal{P}$ such that either

- $\text{body}^+(r) \subseteq Y \wedge \text{body}^-(r) \cap T = \emptyset$ and $\text{head}(r) \notin Y$ **OR**
- $\text{body}^+(r) \subseteq X \wedge \text{body}^-(r) \cap P = \emptyset$ and $\text{head}(r) \notin X$

3 Approximation Fixpoint Theory

We can think of a three-valued interpretation (T, P) as an approximation on the set of true atoms. T is a lower bound and P is the upper bound.

Definition 3.1. An approximator is a fixpoint operator on the complete lattice $\langle \wp(\mathcal{L})^2, \preceq_p \rangle$ (called a bilattice)

Given a function $f(T, P) : S^2 \rightarrow S^2$, we define two separate functions

$$\begin{aligned} f(\cdot, P)_1 : S &\rightarrow S \\ f(T, \cdot)_2 : S &\rightarrow S \end{aligned}$$

such that

$$f(T, P) = \left((f(\cdot, P)_1)(T), \right. \\ \left. (f(T, \cdot)_2)(P) \right)$$

Definition 3.2. Given an approximator $\Phi(T, P)$ the stable revision operator is defined as follows

$$S(T, P) = (\mathbf{lfp}(\Phi(\cdot, P)_1), \mathbf{lfp}(\Phi(T, \cdot)_2))$$

Note: the \mathbf{lfp} is applied to a unary operator, thus it's the least fixpoint of the lattice $\langle \wp(\mathcal{L}), \subseteq \rangle$ whose least element is \emptyset .

3.1 An Approximator for Partial Stable Semantics

$$\Gamma(T, P) := \{head(r) \mid r \in \mathcal{P}, T \subseteq body^+(r), body^-(r) \cap P = \emptyset\}$$

$$\Gamma(P, T) := \{head(r) \mid r \in \mathcal{P}, P \subseteq body^+(r), body^-(r) \cap T = \emptyset\}$$

$$\Phi(T, P) := (\Gamma(T, P), \Gamma(P, T))$$

Without stable revision where $(T, P) = (\{a, b\}, \{a, b\})$

$$\Phi(T, P) = (\{a, b\}, \{a, b\}) \text{ (fixpoint reached)}$$

With stable revision

$$S(T, P) = (\emptyset, \emptyset) \text{ (fixpoint reached)}$$

4 The Polynomial Heirarchy

Intuitive definitions of NP

- the class of problems which have algorithms that can verify solutions in polynomial time.
- A problem that can be solved by reducing it to a SAT expression of the form

$$\exists(a \vee b \vee c) \wedge (\neg a \vee \neg d \vee c) \wedge \dots$$

- (Alternating turing machine) A problem that is solved by some path of an algorithm that is allowed to branch is parallel

Intuitive definitions of NP^{NP} a.k.a. Σ_2^P

- the class of problems which have algorithms that can verify solutions in NP time.
- A problem that can be solved by reducing it to a SAT expression of the form

$$\exists c, \forall a, b, (a \vee b \vee c) \wedge (\neg a \vee \neg d \vee c) \wedge \dots$$

- (Alternating Turing machine) A problem that is solved by some path of an algorithm that is allowed to branch in parallel. A branch is allowed to switch to “ALL” mode only once and require that all subsequent forks return success