Summary of Important Definitions

May 11th

1 Lattice Theory

Definition 1.1. *A* partial order $\preceq$ *is a relation over a set* S *such that for every triple of elements* $x, y, z \in S$ *the following hold*

- *(reflexivity)* $x \preceq x$
- *(antisymmetry)* $(x \preceq y \wedge y \preceq x) \Rightarrow x = y$
- *(transitivity)* $(x \preceq y \wedge y \preceq z) \Rightarrow (x \preceq z)$

Definition 1.2. *Given a partial order* $\preceq$ *over a set* S *and a subset* $X \subseteq S$*, a* lower bound *of* X *(resp. an* upper bound *of* X*) is an element* $x \in S$ *(Note that it may be the case that* $x \notin X$*) such that*

- $\forall y \in X, \boxed{x} \preceq y$
- *(resp.* $\forall y \in X, \boxed{y} \preceq x$*)*

The greatest lower bound *of a set* $X \subseteq S$ *(denoted as* $\textbf{\textit{glb}}(X)$*) is a unique upperbound of the set of all lowerbounds of* X*. The* least upper bound *of a set* $X \subseteq S$ *(denoted as* $\textbf{\textit{lub}}(X)$*) is a unique lowerbound of the set of all upperbounds of* X*. In general,* $\textbf{\textit{lub}}(X)$ *and* $\textbf{\textit{glb}}(X)$ *may not exist.*

Definition 1.3. *A (complete) lattice* $\langle \mathcal{L}, \preceq \rangle$ *is a set of elements* $\mathcal{L}$ *and a partial order* $\preceq$ *over* $\mathcal{L}$ *such that for any set* $S \subseteq \mathcal{L}$

- $\textbf{\textit{lub}}(X)$ *and* $\textbf{\textit{glb}}(X)$ *exist and are unique.*

1.1 Fixpoint Operators

Definition 1.4. *Given a complete lattice* $\langle \mathcal{L}, \preceq \rangle$*, a fixpoint operator over the lattice is a function* $\Phi : \mathcal{L} \to \mathcal{L}$ *that is* $\preceq$-monotone

- Φ *is* $\preceq$*-monotone if for all* $x, y \in \mathcal{L}$

$$x \preceq y \Rightarrow \Phi(x) \preceq \Phi(y)$$

(Note: this does not mean the function is inflationary, i.e., $x \preceq \Phi(x)$ *may not hold)*

A fixpoint *is an element* $x \in \mathcal{L}$ *s.t.* $\Phi(x) = x$*.*

Theorem 1.1 (Knaster-Tarski)**.** *The set of all fixpoints of a fixpoint operator* Φ *on a complete lattice is itself a complete lattice. The least element of this new lattice exists and is called the least fixpoint (denoted as* $\mathbf{lfp}\ \Phi$*)*

Some intuitions about lattices

- The entire lattice has a biggest element $\mathbf{lub}\ (\mathcal{L}) = \top$ and a smallest element $\mathbf{glb}\ (\mathcal{L}) = \bot$
- When a lattice has a finite height (or finite domain). The least fixed point of a fixpoint operator can be computed by iteratively applying the fixpoint operator to $\bot$
- An operator may return an element that is not comparable to the input, however, after a comparable element is returned (either greater or less than) that comparability and direction are maintained for all subsequent iterations.
- Further, because $\bot$ is less than all elements in the lattice, it is always the case that $\bot \preceq \Phi(\bot)$

2 Partial Stable Model Semantics

Definition 2.1. *A (ground and normal)* answer set program $\mathcal{P}$ *is a set of rules where each rule* r *is of the form*

$$h \leftarrow a_0,\ a_1,\ \ldots,\ a_n,\ \mathbf{not}\, b_0,\ \mathbf{not}\, b_1,\ \ldots,\ \mathbf{not}\, b_k$$

where we define the following shorthand for a rule $r \in \mathcal{P}$

$$\begin{aligned} head(r) &= h \\ body^+(r) &= \{a_0,\ a_1,\ \ldots,\ a_n\} \\ body^-(r) &= \{b_0,\ b_1,\ \ldots,\ b_k\} \end{aligned}$$

Definition 2.2. *A two-valued interpretation* I *of a program* $\mathcal{P}$ *is a set of atoms that appear in* $\mathcal{P}$*.*

Definition 2.3. *An interpretation* I *is a* model *of a program* $\mathcal{P}$ *if for each rule* $r \in \mathcal{P}$

- *If* $body^+(r) \subseteq I$ *and* $body^-(r) \cap I = \emptyset$ *then* $head(r) \in I$*.*

Definition 2.4. *An interpretation* I *is a* stable model *of a program* $\mathcal{P}$ *if* I *is a model of* $\mathcal{P}$ ***and*** *for every interpretation* $I' \subseteq I$ *there exists a rule* $r \in \mathcal{P}$ *such that*

- $body^+(r) \subseteq I'$, $body^-(r) \cap I \neq \emptyset$ *(Note that this is* I *and not* I'*) and* $head(r) \notin I'$

Definition 2.5. *A three-valued interpretation* (T, P) *of a program* $\mathcal{P}$ *is a pair of sets of atoms such that* $T \subseteq P$*. The* $\boxed{\textit{truth-ordering}}$ *respects* $\mathbf{f} < \mathbf{u} < \mathbf{t}$ *and is defined for two three-valued interpretations* (T, P) *and* (X, Y) *as follows.*

$$(T, P) \preceq_t (X, Y) \text{ iff } T \subseteq X \wedge P \subseteq Y$$

The $\boxed{\textit{precision-ordering}}$ *respects the partial order* $\mathbf{u} < \mathbf{t}$*,* $\mathbf{u} < \mathbf{f}$ *and is defined for two three-valued interpretations* (T, P) *and* (X, Y) *as follows.*

$$(T, P) \preceq_p (X, Y) \text{ iff } T \subseteq X \wedge \boxed{Y \subseteq P}$$

Definition 2.6. *A three-valued interpretation* (T, P) *is a* model *of a program* $\mathcal{P}$ *if for each rule* $r \in \mathcal{P}$

- $body(r) \subseteq P \wedge body^-(r) \cap T = \emptyset$ *implies* $head(r) \in P$*, and*
- $body(r) \subseteq T \wedge body^-(r) \cap P = \emptyset$ *implies* $head(r) \in T$*.*

Definition 2.7. *A three-valued interpretation* (T, P) *is a* stable model *of a program* $\mathcal{P}$ *if it is a model of* $\mathcal{P}$ *and if for every three-valued interpretation* (X, Y) *such that* $(X, Y) \preceq_t (T, P)$ *there exists a rule* $r \in \mathcal{P}$ *such that either*

- $body^+(r) \subseteq Y \wedge body^-(r) \cap T = \emptyset$ *and* $head(r) \notin Y$ ***OR***
- $body^+(r) \subseteq X \wedge body^-(r) \cap P = \emptyset$ *and* $head(r) \notin X$

3 Approximation Fixpoint Theory

We can think of a three-valued interpretation (T, P) as an approximation on the set of true atoms. T is a lower bound and P is the upper bound.

Definition 3.1. *An* approximator *is a fixpoint operator on the complete lattice* $\langle \wp(\mathcal{L})^2, \preceq_p \rangle$ *(called a bilattice)*

Given a function $f(T, P) : S^2 \to S^2$, we define two separate functions

$$f(\cdot, P)_1 : \; S \to S$$
$$f(T, \cdot)_2 : \; S \to S$$

such that

$$f(T, P) = \Big((f(\cdot, P)_1)(T), \quad (f(T, \cdot)_2)(P) \Big)$$

Definition 3.2. *Given an approximator* $\Phi(T, P)$ *the stable revision operator is defined as follows*

$$S(T, P) = (\mathbf{lfp}(\Phi(\cdot, P)_1), \; \mathbf{lfp}(\Phi(T, \cdot)_2))$$

Note: the **lfp** *is applied to a unary operator, thus it's the least fixpoint of the lattice* $\langle \wp(\mathcal{L}), \subseteq \rangle$ *whose least element is* $\emptyset$*.*

3.1 An Approximator for Partial Stable Semantics

$$\Gamma(T,P) \coloneqq \{head(r) \mid r \in \mathcal{P}, T \subseteq body^+(r), body^-(r) \cap P = \emptyset\}$$
$$\Gamma(P,T) \coloneqq \{head(r) \mid r \in \mathcal{P}, P \subseteq body^+(r), body^-(r) \cap T = \emptyset\}$$
$$\Phi(T,P) \coloneqq \Big(\Gamma(T,P), \Gamma(P,T)\Big)$$

Without stable revision where $(T,P) = (\{a,b\},\{a,b\})$

$$\Phi(T,P) = (\{a,b\},\{a,b\}) \text{ (fixpoint reached)}$$

With stable revision

$$S(T,P) = (\emptyset,\emptyset) \text{ (fixpoint reached)}$$

Proposition 3.1. *Given a poset $\langle \mathcal{L}, \leq \rangle$, an operator $o : \mathcal{L} \to \mathcal{L}$ is $\leq$-monotone iff it is $\geq$-monotone*

Proof. ($\Rightarrow$) Initially, we have $\forall x,y \in \mathcal{L} : x \leq y \Rightarrow o(x) \leq o(y)$. Let $a,b \in \mathcal{L}$ such that $a \geq b$. We have $b \leq a$ and can apply our initial assumption to get $o(b) \leq o(a)$. This gives us $o(a) \geq o(b)$. We can generalize to obtain $\forall x,y \in \mathcal{L} : x \geq y \Rightarrow o(x) \geq o(y)$. ($\Leftarrow$) Proof is more or less the same □

Proposition 3.2. *An operator $A : L^2 \to L^2$ is symmetric and monotone with respect to both $\leq_i$ and $\leq$ if and only if there is a monotone operator $O : L \to L$ such that for every $x,y \in L$, $A(x,y) = (O(x), O(y))$*

Proof. ($\Rightarrow$) From Proposition 5 and 6 we have for any $x \in L$, $A_1(\cdot,x)$ and $A_1(x,\cdot)$ are monotone and $A_1(x,\cdot)$ is antimonotone. By Proposition 2, $A_1(x,\cdot)$ is constant, denote this constant as the function $O(x)$. By the symmetric condition, we have $A_1(x,\cdot) = A_2(\cdot,x)$, thus $A(x,y) = (O(x), O(y))$. It follows from the monotonicity of A that O is monotone.

($\Leftarrow$) Clearly A is symmetric, and $\leq$-monotone (Given that O is $\leq$-monotone). Using proposition 3.1 (above) $O(x)$ is $\geq$-monotone, thus A is $\leq_i$-monotone as well. □

4 The Polynomial Heirarchy

Intuitive definitions of NP

- the class of problems which have algorithms that can verify solutions in polynomial time.
- A problem that can be solved by reducing it to a SAT expression of the form

$$\exists (a \vee b \vee c) \wedge (\neg a \vee \neg d \vee c) \ \wedge \cdots$$

- (Alternating turing machine) A problem that is solved by some path of an algorithm that is allowed to branch is parallel

Intuitive definitions of $\mathsf{NP}^{\mathsf{NP}}$ a.k.a. Σ_2^P

- the class of problems which have algorithms that can verify solutions in NP time.
- A problem that can be solved by reducing it to a SAT expression of the form

$$\exists c,\ \forall a,\ b,\ (a \vee b \vee c) \wedge (\neg a \vee \neg d \vee c)\ \wedge \cdots$$

- (Alternating Turing machine) A problem that is solved by some path of an algorithm that is allowed to branch in parallel. A branch is allowed to switch to "ALL" mode only once and require that all subsequent forks return success