# Summary of Important Definitions

May 11th

## 1 Lattice Theory

**Definition 1.1.** *A* partial order $\preceq$ *is a relation over a set* $S$ *such that for every triple of elements* $x, y, z \in S$ *the following hold*

- *(reflexivity)* $x \preceq x$
- *(antisymmetry)* $(x \preceq y \wedge y \preceq x) \Rightarrow x = y$
- *(transitivity)* $(x \preceq y \wedge y \preceq z) \Rightarrow (x \preceq z)$

**Definition 1.2.** *Given a partial order* $\preceq$ *over a set* $S$ *and a subset* $X \subseteq S$*, a* lower bound *of* $X$ *(resp. an* upper bound *of* $X$*) is an element* $x \in S$ *(Note that it may be the case that* $x \notin X$*) such that*

- $\forall y \in X, \boxed{x} \preceq y$
- *(resp.* $\forall y \in X, \boxed{y} \preceq x$*)*

*The* greatest lower bound *of a set* $X \subseteq S$ *(denoted as* $\textbf{\textit{glb}}(X)$*) is a unique upperbound of the set of all lowerbounds of* $X$*. The* least upper bound *of a set* $X \subseteq S$ *(denoted as* $\textbf{\textit{lub}}(X)$*) is a unique lowerbound of the set of all upperbounds of* $X$*. In general,* $\textbf{\textit{lub}}(X)$ *and* $\textbf{\textit{glb}}(X)$ *may not exist.*

**Definition 1.3.** *A (complete) lattice* $\langle \mathcal{L}, \preceq \rangle$ *is a set of elements* $\mathcal{L}$ *and a partial order* $\preceq$ *over* $\mathcal{L}$ *such that for any set* $S \subseteq \mathcal{L}$

- $\textbf{\textit{lub}}(X)$ *and* $\textbf{\textit{glb}}(X)$ *exist and are unique.*

### 1.1 Fixpoint Operators

**Definition 1.4.** *Given a complete lattice* $\langle \mathcal{L}, \preceq \rangle$*, a fixpoint operator over the lattice is a function* $\Phi : \mathcal{L} \to \mathcal{L}$ *that is* $\preceq$-monotone

- $\Phi$ *is* $\preceq$*-monotone if for all* $x, y \in \mathcal{L}$

$$x \preceq y \Rightarrow \Phi(x) \preceq \Phi(y)$$

  *(Note: this does not mean the function is inflationary, i.e.,* $x \preceq \Phi(x)$ *may not hold)*

*A* fixpoint *is an element $x \in \mathcal{L}$ s.t. $\Phi(x) = x$.*

**Theorem 1.1** (Knaster-Tarski)**.** *The set of all fixpoints of a fixpoint operator $\Phi$ on a complete lattice is itself a complete lattice. The least element of this new lattice exists and is called the least fixpoint (denoted as* **lfp** $\Phi$*)*

Some intuitions about lattices

- The entire lattice has a biggest element **lub** $(\mathcal{L}) = \top$ and a smallest element **glb** $(\mathcal{L}) = \bot$
- When a lattice has a finite height (or finite domain). The least fixed point of a fixpoint operator can be computed by iteratively applying the fixpoint operator to $\bot$
- An operator may return an element that is not comparable to the input, however, after a comparable element is returned (either greater or less than) that comparability and direction are maintained for all subsequent iterations.
- Further, because $\bot$ is less than all elements in the lattice, it is always the case that $\bot \preceq \Phi(\bot)$

# 2 Partial Stable Model Semantics

**Definition 2.1.** *A (ground and normal)* answer set program $\mathcal{P}$ *is a set of rules where each rule $r$ is of the form*

$$h \leftarrow a_0,\ a_1,\ \ldots,\ a_n,\ \textbf{not}\, b_0,\ \textbf{not}\, b_1,\ \ldots,\ \textbf{not}\, b_k$$

*where we define the following shorthand for a rule $r \in \mathcal{P}$*

$$\begin{aligned} head(r) &= h \\ body^+(r) &= \{a_0,\ a_1,\ \ldots,\ a_n\} \\ body^-(r) &= \{b_0,\ b_1,\ \ldots,\ b_k\} \end{aligned}$$

**Definition 2.2.** *A two-valued interpretation $I$ of a program $\mathcal{P}$ is a set of atoms that appear in $\mathcal{P}$.*

**Definition 2.3.** *An interpretation $I$ is a* model *of a program $\mathcal{P}$ if for each rule $r \in \mathcal{P}$*

- *If $body^+(r) \subseteq I$ and $body^-(r) \cap I = \emptyset$ then $head(r) \in I$.*

**Definition 2.4.** *An interpretation $I$ is a* stable model *of a program $\mathcal{P}$ if $I$ is a model of $\mathcal{P}$* ***and*** *for every interpretation $I' \subseteq I$ there exists a rule $r \in \mathcal{P}$ such that*

- *$body^+(r) \subseteq I'$, $body^-(r) \cap I \neq \emptyset$ (Note that this is $I$ and not $I'$) and $head(r) \notin I'$*

**Definition 2.5.** *A three-valued interpretation* $(T, P)$ *of a program* $\mathcal{P}$ *is a pair of sets of atoms such that* $T \subseteq P$*. The* *truth-ordering* *respects* $\mathbf{f} < \mathbf{u} < \mathbf{t}$ *and is defined for two three-valued interpretations* $(T, P)$ *and* $(X, Y)$ *as follows.*

$$(T, P) \preceq_t (X, Y) \text{ iff } T \subseteq X \wedge P \subseteq Y$$

*The* *precision-ordering* *respects the partial order* $\mathbf{u} < \mathbf{t}$, $\mathbf{u} < \mathbf{f}$ *and is defined for two three-valued interpretations* $(T, P)$ *and* $(X, Y)$ *as follows.*

$$(T, P) \preceq_p (X, Y) \text{ iff } T \subseteq X \wedge \boxed{Y \subseteq P}$$

**Definition 2.6.** *A three-valued interpretation* $(T, P)$ *is a* model *of a program* $\mathcal{P}$ *if for each rule* $r \in \mathcal{P}$

- $body(r) \subseteq P \wedge body^-(r) \cap T = \emptyset$ *implies* $head(r) \in P$*, and*
- $body(r) \subseteq T \wedge body^-(r) \cap P = \emptyset$ *implies* $head(r) \in T$*.*

**Definition 2.7.** *A three-valued interpretation* $(T, P)$ *is a* stable model *of a program* $\mathcal{P}$ *if it is a model of* $\mathcal{P}$ *and if for every three-valued interpretation* $(X, Y)$ *such that* $(X, Y) \preceq_t (T, P)$ *there exists a rule* $r \in \mathcal{P}$ *such that either*

- $body^+(r) \subseteq Y \wedge body^-(r) \cap T = \emptyset$ *and* $head(r) \notin Y$ ***OR***
- $body^+(r) \subseteq X \wedge body^-(r) \cap P = \emptyset$ *and* $head(r) \notin X$

# 3 Approximation Fixpoint Theory

We can think of a three-valued interpretation $(T, P)$ as an approximation on the set of true atoms. $T$ is a lower bound and $P$ is the upper bound.

**Definition 3.1.** *An* approximator *is a fixpoint operator on the complete lattice* $\langle \wp(\mathcal{L})^2, \preceq_p \rangle$ *(called a bilattice)*

Given a function $f(T, P) : S^2 \to S^2$, we define two separate functions

$$f(\cdot, P)_1 : \ S \to S$$
$$f(T, \cdot)_2 : \ S \to S$$

such that

$$f(T, P) = \Big( (f(\cdot, P)_1)(T), \\ (f(T, \cdot)_2)(P) \Big)$$

**Definition 3.2.** *Given an approximator* $\Phi(T, P)$ *the stable revision operator is defined as follows*

$$S(T, P) = (\mathbf{lfp}(\Phi(\cdot, P)_1), \ \mathbf{lfp}(\Phi(T, \cdot)_2))$$

*Note: the* **lfp** *is applied to a unary operator, thus it's the least fixpoint of the lattice* $\langle \wp(\mathcal{L}), \subseteq \rangle$ *whose least element is* $\emptyset$*.*

### 3.1 An Approximator for Partial Stable Semantics

$$\Gamma(T, P) \coloneqq \{head(r) \mid r \in \mathcal{P}, T \subseteq body^{+}(r), body^{-}(r) \cap P = \emptyset\}$$
$$\Gamma(P, T) \coloneqq \{head(r) \mid r \in \mathcal{P}, P \subseteq body^{+}(r), body^{-}(r) \cap T = \emptyset\}$$
$$\Phi(T, P) \coloneqq \Big(\Gamma(T, P), \Gamma(P, T)\Big)$$

# 4 The Polynomial Heirarchy

Intuitive definitions of $\mathsf{NP}$

- the class of problems which have algorithms that can verify solutions in polynomial time.
- A problem that can be solved by reducing it to a SAT expression of the form

$$\exists (a \vee b \vee c) \wedge (\neg a \vee \neg d \vee c) \ \wedge \cdots$$

- (Alternating turing machine) A problem that is solved by some path of an algorithm that is allowed to branch is parallel

Intuitive definitions of $\mathsf{NP}^{\mathsf{NP}}$ a.k.a. $\Sigma_2^P$

- the class of problems which have algorithms that can verify solutions in $\mathsf{NP}$ time.
- A problem that can be solved by reducing it to a SAT expression of the form

$$\exists c,\ \forall a,\ b,\ (a \vee b \vee c) \wedge (\neg a \vee \neg d \vee c) \ \wedge \cdots$$

- (Alternating Turing machine) A problem that is solved by some path of an algorithm that is allowed to branch in parallel. A branch is allowed to switch to "ALL" mode only once and require that all subsequent forks return success